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Note

Eppstein's bound on intersecting triangles revisited

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ABSTRACT

Let S be a set of n points in the plane, and let T be a set of m triangles with vertices in S. Then there exists a point in the plane contained in $\Omega(m^3/(n^6\log^2 n))$ triangles of T. Eppstein [D. Eppstein, Improved bounds for intersecting triangles and halving planes, J. Combin. Theory Ser. A 62 (1993) 176–182] gave a proof of this claim, but there is a problem with his proof. Here we provide a correct proof by slightly modifying Eppstein's argument.

1. Introduction

Let S be a set of n points in the plane in general position (no three points on a line), and let T be a set of $m \leqslant \binom{n}{3}$ triangles with vertices in S. Aronov et al. [2] showed that there always exists a point in the plane contained in the interior of

$$\Omega\left(\frac{m^3}{n^6\log^5 n}\right) \tag{1}$$

triangles of T. Eppstein [5] subsequently claimed to have improved this bound to

$$\Omega\left(\frac{m^3}{n^6\log^2 n}\right). \tag{2}$$

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There is a problem in Eppstein's proof, however.[3] In this note we provide a correct proof of (2), by slightly modifying Eppstein's argument.

1.1. The Second Selection Lemma and k-sets

The above result is the special case $d = 2$ of the following lemma (called the *Second Selection Lemma* in [6]), whose proof was put together by Bárány et al. [3], Alon et al. [1], and Živaljević and Vrećica [8]:

Lemma 1. *If S is an n-point set in $\mathbb{R}^d$ and T is a family of $m \leqslant \binom{n}{d+1}$ d-simplices spanned by S, then there exists a point $p \in \mathbb{R}^d$ contained in at least*

$$c_d\left(\frac{m}{n^{d+1}}\right)^{s_d} n^{d+1} \tag{3}$$

simplices of T, for some constants c_d and s_d that depend only on d.

(Note that $m/n^{d+1} = O(1)$, so the smaller the constant s_d, the stronger the bound.) Thus, for $d = 2$ the constant s_2 in (3) can be taken arbitrarily close to 3. The general proof of Lemma 1 gives very large bounds for s_d; roughly $s_d \approx (4d+1)^{d+1}$.

The main motivation for the Second Selection Lemma is deriving upper bounds for the maximum number of *k-sets* of an n-point set in $\mathbb{R}^d$; see [6, Chapter 11] for the definition and details.

2. The proof

We assume that $m = \Omega(n^2 \log^{2/3} n)$, since otherwise the bound (2) is trivial. The proof, like the proof of the previous bound (1), relies on the following two one-dimensional *selection lemmas* [2]:

Lemma 2 *(Unweighted Selection Lemma). Let V be a set of n points on the real line, and let E be a set of m distinct intervals with endpoints in V. Then there exists a point x lying in the interior of $\Omega(m^2/n^2)$ intervals of E.*

Lemma 3 *(Weighted Selection Lemma). Let V be a set of n points on the real line, and let E be a* multiset *of m intervals with endpoints in V. Then there exists a multiset $E' \subseteq E$ of m' intervals, having as endpoints a subset $V' \subseteq V$ of n' points, such that all the intervals of E' contain a common point x in their interior, and such that*

$$\frac{m'}{n'} = \Omega\left(\frac{m}{n \log n}\right).$$

The proof of the desired bound (2) proceeds as follows:

Assume without loss of generality that no two points of S have the same x-coordinate. For each triangle in T define its *base* to be the edge with the longest x-projection. For each pair of points $a, b \in S$, let T_{ab} be the set of triangles in T that have ab as base, and let $m_{ab} = |T_{ab}|$. (Thus, $\sum_{ab} m_{ab} = m$.)

Discard all sets T_{ab} for which $m_{ab} < m/n^2$. We discarded at most $\binom{n}{2} m/n^2 < m/2$ triangles, so we are left with a subset T' of at least $m/2$ triangles, such that either $m_{ab} = 0$ or $m_{ab} \geqslant m/n^2$ for each base ab.[4]

Partition the bases into a logarithmic number of subsets $E_1, E_2, \ldots, E_k$ for $k = \log_4(n^3/m)$, so that each E_j contains all the bases ab for which

$$\frac{4^{j-1}m}{n^2} \leqslant m_{ab} < \frac{4^j m}{n^2}. \tag{4}$$

[3] The very last sentence in the proof of Theorem 4 (Section 4) in [5] reads: "So $\epsilon = 1/2^{i+1}$, and $x = m\epsilon/y = O(m/8^i)$, from which it follows that $x/\epsilon^3 = O(n^2)$." This is patently false, since what actually follows is that $x/\epsilon^3 = O(m)$, and the entire argument falls through.

[4] This critical discarding step is missing in [5], and that is why the proof there does not work.

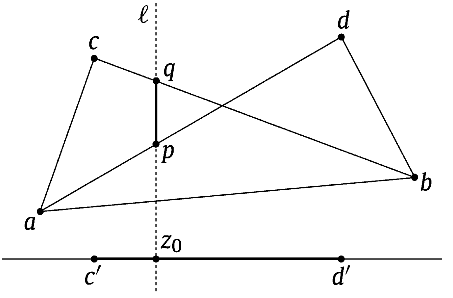

Fig. 1. Pairing two triangles with a common base.

Let $T_j = \bigcup_{ab \in E_j} T_{ab}$ denote the set of triangles with bases in E_j, and $m_j = |T_j|$ denote their number. There must exist an index j for which

$$m_j \geqslant 2^{-(j+1)} m,$$

since otherwise the total number of triangles in T' would be less than $m/2$. From now on we fix this j, and work only with the bases in E_j and the triangles in T_j.

For each pair of triangles abc, abd having the same base $ab \in E_j$, project the segment cd into the x-axis, obtaining segment $c'd'$. We thus obtain a multiset M_0 of horizontal segments, with

$$|M_0| \geqslant \frac{m_j}{2}\left(\frac{4^{j-1} m}{n^2} - 1\right) = \Omega\left(\frac{2^j m^2}{n^2}\right).$$

(Each of the m_j triangles in T_j is paired with all other triangles sharing the same base, and each such pair is counted twice.)

We now apply the Weighted Selection Lemma (Lemma 3) to M_0, obtaining a multiset M_1 of segments delimited by n_1 distinct endpoints, all segments containing some point z_0 in their interior, with

$$\frac{|M_1|}{n_1} = \Omega\left(\frac{|M_0|}{n \log n}\right) = \Omega\left(\frac{2^j m^2}{n^3 \log n}\right).$$

Let ℓ be the vertical line passing through z_0. For each horizontal segment $c'd' \in M_1$, each of its (possibly multiple) instances in M_1 originates from a pair of triangles abc, abd, where points a and c lie to the left of ℓ, and points b and d lie to the right of ℓ. Let p be the intersection of ℓ with ad, and let q be the intersection of ℓ with bc. Then, pq is a vertical segment along ℓ, contained in the union of the triangles abc, abd (see Fig. 1). Let M_2 be the set of all these segments pq for all $c'd' \in M_1$.

Note that the vertical segments in M_2 are all distinct, since each such segment pq uniquely determines the originating points a, b, c, d (assuming z_0 was chosen in general position).

Let n_2 be the number of endpoints of the segments in M_2. We have $n_2 \leqslant nn_1$, since each endpoint (such as p) is uniquely determined by one of n_1 "inner" vertices (such as d) and one of at most n "outer" vertices (such as a).

Next, apply the Unweighted Selection Lemma (Lemma 2) to M_2, obtaining a point $x_0 \in \ell$ that is contained in

$$\Omega\left(\frac{|M_2|^2}{n_2^2}\right) = \Omega\left(\frac{1}{n^2}\left(\frac{|M_1|}{n_1}\right)^2\right) = \Omega\left(\frac{4^j m^4}{n^8 \log^2 n}\right)$$

segments in M_2. Thus, x_0 is contained in at least these many *unions of pairs of triangles* of T_j. But by (4), each triangle in T_j participates in at most $4^j m/n^2$ pairs. Therefore, x_0 is contained in

$$\Omega\left(\frac{m^3}{n^6 \log^2 n}\right)$$

triangles of T_j.

3. Discussion

Eppstein [5] also showed that there always exists a point in $\mathbb{R}^2$ contained in $\Omega(m/n)$ triangles of T. This latter bound is stronger than (2) for small m, namely for $m = O(n^{5/2} \log n)$.

On the other hand, as Eppstein also showed [5], for every n-point set S in general position and every $m = \Omega(n^2)$, $m \leqslant \binom{n}{3}$, there exists a set T of m triangles with vertices in S, such that no point in the plane is contained in more than $O(m^2/n^3)$ triangles of T. Thus, with the current lack of any better lower bound, the bound (2) appears to be far from tight. Even achieving a lower bound of $\Omega(m^3/n^6)$, without any logarithmic factors, is a major challenge still unresolved.

It is known, however, that if S is a set of n points in $\mathbb{R}^3$ in general position (no four points on a plane), and T is a set of m triangles spanned by S, then there exists a *line* (in fact, a line spanned by two points of S) that intersects the interior of $\Omega(m^3/n^6)$ triangles of T; see [4] and [7] for two different proofs of this.

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